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Perturbative Symmetries on Noncommutative Spaces

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Abstract

Perturbative deformations of symmetry structures on noncommutative spaces are studied in view of noncommutative quantum field theories. The rigidity of enveloping algebras of semi-simple Lie algebras with respect to formal deformations is reviewed in the context of star products. It is shown that rigidity of symmetry algebras extends to rigidity of the action of the symmetry on the space. This implies that the noncommutative spaces considered can be realized as star products by particular ordering prescriptions which are compatible with the symmetry. These symmetry preserving ordering prescriptions are calculated for the quantum plane and four-dimensional quantum Euclidean space. Using these ordering prescriptions greatly facilitates the construction of invariant Lagrangians for quantum field theory on noncommutative spaces with a deformed symmetry.

1 Introduction

If the geometry of a physical space is noncommutative at energies accessible by current accelerators [1] the noncommutativity can only be small. This suggests to describe noncommutative spaces as perturbative deformations of ordinary, commutative spaces. If such a small deformation is to have controllably small effects, it must depend in some sense smoothly on a deformation parameter. Ideally, the deformation of a physical quantity which results from a perturbation of the geometry is given by a convergent perturbation series in powers of the deformation parameter. But even if convergence in this strong sense is rigorously not possible, as for quantum field theory or deformation quantization, the perturbation series may still be useful in an algorithmic or algebraic sense.

The algebraic aspects of a deformation can be separated from the analytic questions of continuity and convergence by considering formal power series. In such a framework, a noncommutative space can be described as formal deformation of the algebra of functions on the space manifold in the sense of Gerstenhaber [2]: The deformed algebra is an algebra over the ring of formal power series in the perturbation parameter, which is in zeroth order isomorphic to the undeformed algebra. As it turns out, formal power series are the natural setting for the construction of gauge theories on general noncommutative geometries, which have found a solid formulation [3, 4] within the framework of deformation quantization. For reviews of noncommutative field theories see [5] and [6].

The perturbative approach to noncommutative field theories, that is, expanding the product of noncommutative quantum fields in the noncommutativity parameters and relating the noncommutative gauge potentials and fields to their ordinary, commutative counterparts via the Seiberg–Witten map has put quantum theories on noncommutative spaces within the range of phenomenological considerations: A minimal noncommutative extension of the standard model was formulated [7] the effects of noncommutative geometry on magnetic and electric moments were studied [8, 9] noncommutative neutrino-photon coupling with possible astrophysical implications was investigated [10] the OPAL collaboration has started looking for noncommutative signatures in electron positron pair annihilation [11] just to name some recent examples. For a review on the phenomenological implications of noncommutative geometry see [12].

As important as the spaces are the symmetries which act on them. Deforming a space algebra which transforms covariantly under a symmetry Lie group will in general break the symmetry. For example, the noncommutative geometry which most papers on noncommutative field theory have considered, where the commutator of the space-time observables $[X_\mu, X_\nu] = \theta_{\mu\nu}$ is a constant antisymmetric matrix, breaks Lorentz symmetry. Physically, this has to be expected as the constant commutator can be viewed as due to a constant background field, in string theory a constant B-field on a D-brane. It could be argued that if the noncommutativity parameters $\theta_{\mu\nu}$ are small, the violation of Lorentz symmetry is only small, too. However, on the level of regularization of loop diagrams the noncommutativity leads to an interdependence of ultra-violet and infra-red cutoff scales [13, 14]. This UV/IR mixing seems to put even large scale Lorentz symmetry and weakened notions of locality of noncommutative quantum field theory into doubt [15]. But as yet, UV/IR mixing was investigated in detail only for the case of constant $\theta_{\mu\nu}$.

The appearance of UV/IR mixing seems to indicate, that the breaking of symmetries which happens when a space is noncommutatively deformed with constant $\theta_{\mu\nu}$ is not under good control. In a self-contained theory it would be reasonable to expect $\theta_{\mu\nu}$ to become itself a field, which transforms covariantly with respect to a perturbative deformation of space-time symmetry. The existence of a deformed symmetry structure would be a big advantage for phenomenological considera-

tions. It would allow to include in the perturbative approach the changes induced by noncommutativity to those physical concepts that are tied to space-time symmetry, such as energy-momentum conservation, Lorentz invariance, independence of in and out states etc.

Another motivation to study noncommutative spaces with deformed symmetries has emerged recently from the attempts to explain the observation [16] of cosmic rays of energy beyond the spectral cutoff (the Greisen–Zatsepin–Kuzmin limit) which is expected due to interaction with the cosmic microwave background. The often proposed explanation of such ultra high energy rays by vacuum dispersion relations, that is, the dependence of the speed of light on the wavelength, was shown by Amelino-Camelia to be reconcilable in principle with the observer independence of the laws of physics [17]. This leads to a deformation of special relativity by the assumption that there is not only an observer invariant velocity but also an observer invariant length, the Planck length, which plays the role of the deformation parameter. This proposition, now called doubly special relativity, has initiated a large number of active studies from both, the mathematical and the phenomenological viewpoint. (For an overview see [18].) Realizations of doubly special relativity can lead to noncommutative deformations of the space-time with a deformed Lorentz symmetry [19].

The purpose of this paper is to study formal perturbative deformations of symmetry structures on noncommutative spaces. On ordinary commutative spaces symmetry structures can be described by Lie algebras or, equivalently, their enveloping algebras. If the Lie algebra is semi-simple as for most interesting cases in physics, the relation of an enveloping algebra to its deformation turns out to be surprisingly simple: The two algebras are isomorphic. More precisely, it can be shown by homological arguments [2] that the enveloping algebra of a semi-simple Lie algebra is *rigid*, that is, over formal power series *any* deformation is isomorphic to the undeformed algebra. We recall this result in theorem 1.

In addition to the symmetry algebra we need the action of the symmetry algebra on the space in order to describe the symmetry structure completely. *A priori*, even though the symmetry algebra is rigid, the action could be truly deformed. However, using the same homological methods as before, we show, that the action is rigid, as well: Over formal power series, the space with an arbitrarily deformed action is isomorphic as module to the space with the usual action of the enveloping Lie algebra by differential operators. This result is stated in theorem 2. In the context of star products vector space isomorphisms between the deformed and the undeformed space are often referred to as ordering prescriptions. In this language theorem 2 shows that there are particular ordering prescriptions which are compatible with the symmetry structure of the space.

Since the widely studied case of constant $\theta_{\mu\nu}$ does not allow for a perturbative deformation of Lorentz symmetry, it cannot serve as example for these general theorems. The standard examples for noncommutative spaces with deformed symmetry structures are quantum spaces [20–22]. They carry a covariant

representation of the Drinfeld–Jimbo deformation [23, 24] of the enveloping symmetry algebra. In fact, rigidity theorem 1 which we review here lies at the core of Drinfeld's pioneering work on quantum enveloping algebras [25, 26].

While from a mathematical point of view isomorphic objects are often identified, an isomorphism can change the physical interpretation of the symmetry structure. Finding the isomorphisms between the deformed and undeformed structures on an algebraic level, however, is a difficult computational problem because the homological methods which are used to prove the rigidity theorems, although elegant, are inherently non-constructive. We will use representation theory to reduce the algebraic problem to matrix calculations. This approach works well for cases, where the representation theory is well understood such as for quantum spaces and quantum algebras.

We will take the quantum plane with its $\mathcal{U}_\hbar(\mathrm{su}_2)$-symmetry as guiding example to demonstrate these representation theoretic methods. The construction of isomorphisms between enveloping algebras and their quantum deformations is text-book material (e.g. Sec. 6.1.3 of [27]). The isomorphism of the module structures has received less attention, but it is of importance for the realization of noncommutative spaces by star products: As in deformation quantization, the multiplication map of a given noncommutative space algebra is often transferred to a commutative function algebra using an ordering prescription, by which the spaces are identified as vector spaces. The rigidity of the module structure as formulated in theorem 2 tells us that there is an ordering prescription which is not only an isomorphism of vector spaces but also of modules. The fact that the deformed and undeformed spaces are isomorphic as modules will only be obscured by most ordering prescriptions, such as the popular normal ordering and the symmetric ordering.

The main result of this paper is the calculation of the symmetry preserving ordering prescription for quantum Euclidean four-space in Eq. (34). The result is expressed in terms of the deformed and undeformed binomial and Clebsch–Gordan coefficients, that is, in terms of basic hypergeometric series. In this sense the representation theoretic approach profits from the computational effort that has gone into the calculation of the q-Clebsch–Gordan coefficients. Trying to redo the calculation which leads to Eq. (34) in a recursive fashion order by order in the deformation parameter, one would quickly learn that q-hypergeometric functions are an extremely efficient way to describe the complex combinatorics of partitions, the reason for which they were first introduced by Euler in the 18th century.

For a rigorous outline of the basics of $\hbar$-adic topologies on algebras in the context of deformation theory, which are used in this article, we refer the reader to [28], Ch. XVI.

2 Space Deformations and Symmetries

2.1 A guiding example

As guiding example for a noncommutatively deformed space with a symmetry structure let us take the $\hbar$-adic quantum plane, the $\hbar$-adic complex algebra with two generators $\hat{x}$, $\hat{y}$ and commutation relation $\hat{x}\hat{y} = \mathrm{e}^{\hbar}\hat{y}\hat{x}$. Expanding this relation in orders of the deformation parameter, we see that in zeroth order the generators $\hat{x}$ and $\hat{y}$ commute. Hence, the polynomial algebra in two variables and the quantum plane,

$$\mathcal{X} := \mathbb{C}[x, y] \quad \text{and} \quad \mathcal{X}_\hbar := \mathbb{C}\langle \hat{x}, \hat{y}\rangle[[\hbar]]/\langle \hat{x}\hat{y} = \mathrm{e}^{\hbar}\hat{y}\hat{x}\rangle\,, \tag{1}$$

are isomorphic modulo $\hbar$. That is, there is an isomorphism of algebras $\xi : \mathcal{X} \to \mathcal{X}_\hbar/\hbar\mathcal{X}_\hbar$ which is defined on the generators as $\xi(x) = \hat{x}$, $\xi(y) = \hat{y}$.

The isomorphism of algebras ξ is *a fortiori* an isomorphism of vector spaces and can be extended to formal power series yielding a $\mathbb{C}[[\hbar]]$-linear isomorphism of $\hbar$-adic vector spaces $\varphi : \mathcal{X}[[\hbar]] \to \mathcal{X}_\hbar$. Such an extension φ, which we will call an ordering prescription, is not unique. For example, the image of the quadratic term xy could be equally defined as $\varphi(xy) := \hat{x}\hat{y}$ (normal ordering) or as $\varphi(xy) := \frac{1}{2}(\hat{x}\hat{y}+\hat{y}\hat{x})$ (symmetric ordering), just to take two popular ordering prescriptions.

Using the ordering prescription, we can transfer the noncommutative multiplication map $\mu_\hbar$ of $\mathcal{X}_\hbar$ to $\mathcal{X}[[\hbar]]$ by requiring

$$\begin{array}{ccc}
\mathcal{X}[[\hbar]]\,\hat{\otimes}\,\mathcal{X}[[\hbar]] & \xrightarrow{\varphi\otimes\varphi} & \mathcal{X}_\hbar\,\hat{\otimes}\,\mathcal{X}_\hbar \\
\big\downarrow{\scriptstyle \mu_\varphi} & & \big\downarrow{\scriptstyle \mu_\hbar} \\
\mathcal{X}[[\hbar]] & \xrightarrow{\varphi} & \mathcal{X}_\hbar
\end{array} \tag{2}$$

to be a commutative diagram, where $\hat{\otimes}$ denotes the topological tensor product. We will call the transferred multiplication map

$$\mu_\varphi := \varphi^{-1} \circ \mu_\hbar \circ (\varphi \otimes \varphi) \tag{3}$$

a star product, denoted by $\mu_\varphi(x \otimes x') \equiv x \star x'$. By construction, the vector space $\mathcal{X}[[\hbar]]$ equipped with this star product is now isomorphic as algebra to $\mathcal{X}_\hbar$, $(\mathcal{X}[[\hbar]], \mu_\varphi) \cong (\mathcal{X}_\hbar, \mu_\hbar)$. While a different ordering prescription φ' will in general yield a different multiplication map $\mu_{\varphi'} \neq \mu_\varphi$, the algebras will be isomorphic, $(\mathcal{X}[[\hbar]], \mu_{\varphi'}) \cong (\mathcal{X}[[\hbar]], \mu_\varphi)$, with $\varphi^{-1} \circ \varphi'$ being an isomorphism.

At this point the construction of the star product seems somewhat vacuous. The reason for transferring the noncommutativity to the ordinary function space is the additional structure on $\mathcal{X}$ which might not be present on $\mathcal{X}_\hbar$, such as a differential calculus, integration, or a symmetry.

The function algebra of the plane can be equipped with a $\mathcal{U}(\mathrm{sl}_2)$ symmetry structure. The Lie algebra sl_2 of the special linear group in two dimensions is

generated by the Cartan–Weyl generators E, H, and F with relations $[H, E] = 2E$, $[H, F] = -2F$, $[E, F] = H$. By definition, the elements of a Lie algebra act on the function space $\mathcal{X}$ as derivations, so we can represent the generators by first order differential operators

$$E = y\partial_x\,, \quad H = y\partial_y - x\partial_x\,, \quad F = x\partial_y\,. \tag{4}$$

This symmetry structure can be deformed together with the deformation of the algebra of the plane into the quantum plane. The deformed symmetry algebra $\mathcal{U}_\hbar(\mathrm{sl}_2)$ is generated by $\hat{E}$, $\hat{H}$, and $\hat{F}$ with relations $[\hat{H}, \hat{E}] = 2\hat{E}$, $[\hat{H}, \hat{F}] = -2\hat{F}$, and

$$[\hat{E}, \hat{F}] = \frac{\mathrm{e}^{\hbar\hat{H}} - \mathrm{e}^{-\hbar\hat{H}}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}}\,. \tag{5}$$

We can define the action of $\hat{E}$, $\hat{H}$, and $\hat{F}$ on the generators $\hat{x}$ and $\hat{y}$ of the quantum plane exactly as for the undeformed case: $\hat{H}$ is diagonal, $\hat{H} \triangleright x = -x$, $\hat{H} \triangleright y = y$, and $\hat{E}$, $\hat{F}$ are ladder operators, $\hat{E} \triangleright x = y$, $\hat{E} \triangleright y = 0$, $\hat{F} \triangleright y = x$, $\hat{F} \triangleright x = 0$. However, while $\hat{H}$ still acts on $\mathcal{X}_h$ as derivation, we have to modify the Leibniz rule for $\hat{E}$ and $\hat{F}$ to

$$\hat{E} \triangleright (\hat{p}_1\hat{p}_2) = (\hat{E} \triangleright \hat{p}_1)(\mathrm{e}^{\hbar\hat{H}} \triangleright \hat{p}_2) + \hat{p}_1(\hat{E} \triangleright \hat{p}_2) \tag{6a}$$

$$\hat{F} \triangleright (\hat{p}_1\hat{p}_2) = (\hat{F} \triangleright \hat{p}_1)\hat{p}_2 + (\mathrm{e}^{-\hbar\hat{H}} \triangleright \hat{p}_1)(\hat{F} \triangleright \hat{p}_2)\,, \tag{6b}$$

for $\hat{p}_1, \hat{p}_2 \in \mathcal{X}_\hbar$. Expanding Eqs. (5) and (6) in $\hbar$, we see that in zeroth order the commutation relations and the Leibniz rule coincide with their undeformed counterparts. This shows that the symmetry structure of the quantum plane is a deformation of the symmetry structure of the plane.

It is natural to ask how the deformed and undeformed symmetries are related. This question can be answered not only for the quantum plane but for the general case.

2.2 The rigidity of symmetry structures

Let $\mathcal{X} = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial function algebra of an n-dimensional space and $\mathcal{X}_\hbar$ be a formal deformation of $\mathcal{X}$ in the sense of Gerstenhaber [2] with deformation parameter $\hbar$. That is, $\mathcal{X}_\hbar$ is an $\hbar$-adic algebra which is isomorphic to $\mathcal{X}$ modulo $\hbar$ as algebra, $\mathcal{X} \cong \mathcal{X}_\hbar/\hbar\mathcal{X}_\hbar$. In other words, setting $\hbar = 0$ in $\mathcal{X}_\hbar$ yields $\mathcal{X}$ as undeformed limit. Clearly, this notion of a formal deformation as "isomorphic modulo $\hbar$" can be extended to other algebraic structures such as a Hopf or a module structure. In addition to this deformation property, we will assume $\mathcal{X}_\hbar$ to be $\mathbb{C}[[\hbar]]$-linearly isomorphic to $\mathcal{X}[[\hbar]]$ as vector space. (Mathematically speaking, this is equivalent to assuming that $\mathcal{X}_\hbar$ is topologically free.) From a computational viewpoint, deformations without this property would have some pathological properties. As before, we will call a $\mathbb{C}[[\hbar]]$-linear isomorphism of vector spaces $\varphi : \mathcal{X}[[\hbar]] \to \mathcal{X}_\hbar$ an ordering prescription.

Let us further assume that there are module structures on $\mathcal{X}$ and $\mathcal{X}_\hbar$, denoting the module maps by

$$\rho : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{X} \to \mathcal{X}\,, \qquad \rho_\hbar : \mathcal{U}_\hbar(\mathfrak{g}) \mathbin{\hat{\otimes}} \mathcal{X}_\hbar \to \mathcal{X}_\hbar\,, \tag{7}$$

where $\mathfrak{g}$ is a semi-simple Lie algebra. We assume that the module structure of the deformed space is a formal deformation of the undeformed module structure, as well. This means, that the symmetry algebra $\mathcal{U}_\hbar(\mathfrak{g})$ is a deformation of $\mathcal{U}(\mathfrak{g})$ and the action $\rho_\hbar$ is a deformation of ρ in the sense of "isomorphic modulo $\hbar$". The deformation property of $\mathcal{U}_\hbar(\mathfrak{g})$ turns out to be surprisingly restrictive:

Theorem 1 (Gerstenhaber–Whitehead). *Let $\mathcal{U}(\mathfrak{g})$ be the enveloping algebra of a semi-simple Lie algebra $\mathfrak{g}$ and $\mathcal{U}_\hbar(\mathfrak{g})$ an $\hbar$-adic algebra which is isomorphic to $\mathcal{U}(\mathfrak{g})$ modulo $\hbar$, $\mathcal{U}_\hbar(\mathfrak{g})/\hbar\mathcal{U}_\hbar(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g})$. Then $\mathcal{U}_\hbar(\mathfrak{g})$ is isomorphic to $\mathcal{U}(\mathfrak{g})[[\hbar]]$ as $\hbar$-adic algebra.*

This theorem tells us that $\mathcal{U}(\mathfrak{g})$ cannot be truly deformed at all. Algebras with this property are called rigid. Gerstenhaber has shown [2] that an algebra is rigid if its second Hochschild cohomology is zero. The second Whitehead lemma implies that the second Hochschild cohomology for the enveloping algebra of a semi-simple Lie algebra is zero, $H^2(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g})) = 0$. Hence, $\mathcal{U}(\mathfrak{g})$ is rigid, which was observed and used by Drinfeld [25,26]. Note that, while the algebra structure of $\mathcal{U}_\hbar(\mathfrak{g})$ is not a true deformation of $\mathcal{U}(\mathfrak{g})$, the deformation of the Hopf structure of $\mathcal{U}_\hbar(\mathfrak{g})$ (the Leibniz rule) is *not* isomorphic to the Hopf structure of $\mathcal{U}(\mathfrak{g})$.

We now turn to the deformation $\rho_\hbar$ of the action ρ. Let $\alpha : \mathcal{U}(\mathfrak{g})[[\hbar]] \to \mathcal{U}_\hbar(\mathfrak{g})$ be the isomorphism of $\hbar$-adic algebras from theorem 1. Let $\varphi : \mathcal{X}[[\hbar]] \to \mathcal{X}_\hbar$ be an ordering prescription. Using the isomorphisms α and φ we can proceed as for the multiplication and transfer the deformed action $\rho_\hbar$ of $\mathcal{U}_\hbar(\mathfrak{g})$ on $\mathcal{X}_\hbar$ to an action of $\mathcal{U}(\mathfrak{g})[[\hbar]]$ on $\mathcal{X}[[\hbar]]$. Requiring the diagram

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g})[[\hbar]] \mathbin{\hat{\otimes}} \mathcal{X}[[\hbar]] & \xrightarrow{\alpha\otimes\varphi} & \mathcal{U}_\hbar(\mathfrak{g}) \mathbin{\hat{\otimes}} \mathcal{X}_\hbar \\ \Big\downarrow{\scriptstyle \rho_{\alpha,\varphi}} & & \Big\downarrow{\scriptstyle \rho_\hbar} \\ \mathcal{X}[[\hbar]] & \xrightarrow{\varphi} & \mathcal{X}_\hbar \end{array} \tag{8}$$

to commute, we have to define the transferred action as

$$\rho_{\alpha,\varphi} = \varphi^{-1} \circ \rho_\hbar \circ (\alpha \otimes \varphi)\,. \tag{9}$$

The assumption that $\rho_\hbar$ is a deformation of ρ in the sense of "isomorphic modulo $\hbar$" can now be conveniently stated as

$$\rho_{\alpha,\varphi} = \rho + \mathcal{O}(\hbar)\,, \tag{10}$$

which can be shown to hold independently of the choice of α and φ. It turns out that the action of the symmetry algebra on $\mathcal{X}$ cannot be truly deformed, either:

Theorem 2. *Let $\rho_\hbar$ and ρ be module maps as in Eq.* (7) *such that $\rho_\hbar$ is a deformation of ρ. Then there is an ordering prescription φ such that for $\rho_{\alpha,\varphi}$ defined as in Eq.* (9) *we have $\rho_{\alpha,\varphi} = \rho$.*

Proof. The first Whitehead lemma, which states that for semi-simple $\mathfrak{g}$ the first Lie algebra cohomology group vanishes, implies that the first Hochschild cohomology of $\mathcal{U}(\mathfrak{g})$ is zero, as well, $H^1(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g})) = 0$.

Consider the undeformed and deformed structure homomorphisms

$$R : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}[[\hbar]]}(\mathcal{X}[[\hbar]]) \qquad R(g)x := \rho(g \otimes x) \tag{11a}$$
$$R_{\alpha,\varphi} : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}[[\hbar]]}(\mathcal{X}[[\hbar]]) \qquad R_{\alpha,\varphi}(g)x := \rho_{\alpha,\varphi}(g \otimes x) \tag{11b}$$

The deformation property now reads $R_{\alpha,\varphi} = R + \mathcal{O}(\hbar)$. The vanishing of the first Hochschild cohomology implies that $R_{\alpha,\varphi}$ and R are related by an inner automorphism (see e.g. [28]). That is, there is an invertible $A \in \mathrm{End}_{\mathbb{C}[[\hbar]]}(\mathcal{X}[[\hbar]])$ such that $R_{\alpha,\varphi}(g) = A\, R(g)\, A^{-1}$.

Now we can define an ordering prescription by $\varphi' := \varphi \circ A$. By definition (9) of the action we get $R_{\alpha,\varphi'}(g) = A^{-1} R_{\alpha,\varphi}(g)\, A = R(g)$. Looking at definition (11) of the structure maps we conclude that $\rho_{\alpha,\varphi'} = \rho$. □

Theorems 1 and 2 are very general. No matter how radical a formal deformation of the symmetry structure may seem, it is always isomorphic to the undeformed symmetry. Most ordering prescriptions, such as the popular normal or symmetric ordering, will only obscure this fact. But how do we find the isomorphism of algebras of theorem 1 and the "good", symmetry preserving ordering prescription of theorem 2? There is no general answer to this question, because the elegant homological methods by which the rigidity theorems can be proved are non-constructive. Nevertheless, we will demonstrate for the quantum plane and quantum Euclidean four-space how representation theory can provide a computational access on this problem.

2.3 Symmetry preserving ordering prescriptions

For each $j \in \frac{1}{2}\mathbb{N}_0$ there is an irreducible spin-j representation of the deformed symmetry algebra $\mathcal{U}_\hbar(\mathrm{sl}_2)$, defined on the generators $\hat{E}$, $\hat{H}$, and $\hat{F}$ by [29]

$$\begin{aligned} \hat{E}|j,m\rangle &= \mathrm{e}^{\hbar(m+1)}\sqrt{[j+m+1][j-m]}\,|j,m+1\rangle \\ \hat{F}|j,m\rangle &= \mathrm{e}^{-\hbar m}\sqrt{[j+m][j-m+1]}\,|j,m-1\rangle \\ \hat{H}|j,m\rangle &= 2m|j,m\rangle\,, \end{aligned} \tag{12}$$

on the $(2j+1)$-dimensional weight basis $\{|j,m\rangle, m = -j, -j+1, \ldots, j\}$. Here, each pair of brackets denotes a quantum number,

$$[a] := \frac{\mathrm{e}^{\hbar a} - \mathrm{e}^{-\hbar a}}{\mathrm{e}^{\hbar} - \mathrm{e}^{-\hbar}}\,, \tag{13}$$

which is an $\hbar$-adic series in the indeterminate a.

The undeformed limit of Eqs. (12) yields the spin-j representation of the undeformed algebra $\mathcal{U}(\mathrm{sl}_2)$ defined on the generators by

$$\begin{aligned} E|j,m\rangle &= \sqrt{(j+m+1)(j-m)}\,|j,m+1\rangle \\ F|j,m\rangle &= \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle \\ H|j,m\rangle &= 2m|j,m\rangle\,. \end{aligned} \tag{14}$$

Let us formally define the operators $M := \frac{1}{2}H$ and

$$J := \tfrac{1}{2}(\sqrt{2C+1}-1)\,, \quad \text{where} \quad C := EF+FE+\tfrac{1}{2}H^2\,, \tag{15}$$

such that $M|j,m\rangle = m|j,m\rangle$ and $J|j,m\rangle = j|j,m\rangle$. Remark, that J is not an element of $\mathcal{U}(\mathfrak{g})$ proper. Now we can define

$$\begin{aligned} \alpha^{-1}(\hat{E}) &:= E\mathrm{e}^{\hbar(M+1)}\sqrt{\frac{[J+M+1][J-M]}{(J+M+1)(J-M)}} \\ \alpha^{-1}(\hat{F}) &:= F\mathrm{e}^{-\hbar M}\sqrt{\frac{[J+M][J-M+1]}{(J+M)(J-M+1)}} \\ \alpha^{-1}(\hat{H}) &:= H \end{aligned} \tag{16}$$

where the right hand sides have to be understood as $\hbar$-adic series with polynomials in J and M as coefficients. Since all expressions involved are symmetric with respect to $J \mapsto -J-1$, the operator J appears in the coefficient polynomials only as polynomial of $2J(J+1) = C$. For example,

$$\alpha^{-1}(\hat{E}) = E\big\{1 + \tfrac{1}{2}(2+H)\hbar + \tfrac{1}{12}[C+(1+H)(5+2H)]\hbar^2\big\} + \mathcal{O}(\hbar^3)\,. \tag{17}$$

We conclude, that the operators defined in Eq. (16) can be viewed as elements of $\mathcal{U}(\mathrm{sl}_2)[[\hbar]]$.

Comparing Eqs. (12) and (14) we see that the operators $\alpha^{-1}(\hat{E})$, $\alpha^{-1}(\hat{F})$, and $\alpha^{-1}(\hat{H})$ have the same irreducible representations as $\hat{E}$, $\hat{F}$, and $\hat{H}$, respectively. Hence, Eqs. (16) define a homomorphism of algebras $\alpha^{-1} : \mathcal{U}_\hbar(\mathrm{sl}_2) \to \mathcal{U}(\mathrm{sl}_2)[[\hbar]]$. Analogously to Eqs. (16) we can construct the inverse of α^{-1}, which shows that α^{-1} is an isomorphism of algebras. Its inverse α is the searched-for isomorphism of theorem 1.

In order to find the symmetry preserving ordering prescription of theorem 2 we want to decompose the plane $\mathcal{X}$ and the quantum plane $\mathcal{X}_\hbar$, as defined in Eq. (1), into irreducible subrepresentations with respect to the $\mathcal{U}(\mathrm{sl}_2)[[\hbar]] \cong \mathcal{U}_\hbar(\mathrm{sl}_2)$ module structure. Define the representation matrices by

$$R^j(g)^m{}_{m'} := \langle j,m|g|j,m'\rangle \quad \text{and} \quad R^j_\hbar(\hat{g})^m{}_{m'} := \langle j,m|\hat{g}|j,m'\rangle \tag{18}$$

for all $g \in \mathcal{U}(\mathrm{sl}_2)$, $\hat{g} \in \mathcal{U}_\hbar(\mathrm{sl}_2)$. Note, that the isomorphism $\alpha : \mathcal{U}(\mathrm{sl}_2)[[\hbar]] \to \mathcal{U}(\mathrm{sl}_2)$ was defined in Eqs. (16) precisely such that

$$R^j(g)^m{}_{m'} = R^j_\hbar(\alpha(g))^m{}_{m'} \,. \tag{19}$$

The basis of $\mathcal{X}_\hbar$ which decomposes the quantum plane into irreducible subrepresentations can be computed to [30]

$$\hat{T}^j_m := \begin{bmatrix} 2j \\ j+m \end{bmatrix}^{\frac{1}{2}}_{q^{-2}} \hat{x}^{j-m} \hat{y}^{j+m} \,, \quad \text{where} \quad \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-2}} := \mathrm{e}^{\hbar k(k-n)} \frac{[n]!}{[n-k]![k]!} \tag{20}$$

is a deformation of the binomial coefficient (sticking to the standard notation of [27]) and $[n]! := [1][2]\cdots[n]$ for natural n. The expression for the basis which decomposes the commutative plane is the undeformed limit

$$T^j_m := \binom{2j}{j+m}^{\frac{1}{2}} x^{j-m} y^{j+m} \,. \tag{21}$$

The action of the symmetry algebras on these bases is

$$\rho_\hbar(\hat{g} \otimes \hat{T}^j_m) = \hat{T}^j_{m'} R^j_\hbar(\hat{g})^{m'}{}_m \quad \text{and} \quad \rho(g \otimes T^j_m) = T^j_{m'} R^j(g)^{m'}{}_m \,, \tag{22}$$

where we sum over repeated indices. Defining an ordering prescription by

$$\varphi(T^j_m) = \hat{T}^j_m \quad \Leftrightarrow \quad \varphi(x^k y^l) = \begin{bmatrix} k+l \\ k \end{bmatrix}^{\frac{1}{2}}_{q^{-2}} \binom{k+l}{k}^{-\frac{1}{2}} \hat{x}^k \hat{y}^l \,, \tag{23}$$

we find that for all $g \in \mathcal{U}(\mathfrak{g})$ we have

$$\begin{aligned} \rho(g \otimes T^j_m) &= T^j_{m'} R^j(g)^{m'}{}_m = (\varphi^{-1} \circ \varphi)\big(T^j_{m'} R^j(g)^{m'}{}_m\big) \\ &= \varphi^{-1}\big(\hat{T}^j_{m'} R^j_\hbar(\alpha(g))^{m'}{}_m\big) = \varphi^{-1}\big(\rho_\hbar(\alpha(g) \otimes \hat{T}^j_{m'})\big) \\ &= (\varphi^{-1} \circ \rho_\hbar \circ [\alpha \otimes \varphi])(g \otimes T^j_{m'}) = \rho_{\alpha,\varphi}(g \otimes T^j_{m'}) \,, \end{aligned} \tag{24}$$

where we used Eqs. (22), (23), (19), and (9). Since $\{T^j_m\}$ is a basis of $\mathcal{X}$, it follows that $\rho_{\alpha,\varphi} = \rho$. Hence, the ordering prescription (23) is the searched-for symmetry preserving ordering of theorem 2.

2.4 Quantum Euclidean four-space

Finally, we want to give the result of the analogous computations for quantum Euclidean four-space.

The algebra of a commutative four-dimensional space is the polynomial algebra generated by its four coordinates. For convenience, the coordinates can be arranged in a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := \left(\begin{smallmatrix} x_0 - \mathrm{i}x_3 & \mathrm{i}x_1 + x_2 \\ \mathrm{i}x_1 - x_2 & x_0 + \mathrm{i}x_3 \end{smallmatrix}\right)$ such that square of the invariant four-length $l^2 := x_0^2 + x_1^2 + x_2^2 + x_3^2$ is given by the determinant. The Euclidean four-space algebra can now be viewed as the algebra of 2×2-matrices matrices $M(2) := \mathbb{C}[a, b, c, d]$.

Quantum Euclidean four-space is defined as the $\hbar$-adic algebra $M_\hbar(2)$ of quantum 2×2-matrices, generated by $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ with relations

$$\begin{gathered}\hat{a}\hat{b} = \mathrm{e}^\hbar \hat{b}\hat{a}\,, \quad \hat{a}\hat{c} = \mathrm{e}^\hbar \hat{c}\hat{a}\,, \quad \hat{b}\hat{d} = \mathrm{e}^\hbar \hat{d}\hat{b}\,, \quad \hat{c}\hat{d} = \mathrm{e}^\hbar \hat{d}\hat{c} \\ \hat{b}\hat{c} = \hat{c}\hat{b}\,, \quad \hat{a}\hat{d} - \hat{d}\hat{a} = (\mathrm{e}^\hbar - \mathrm{e}^{-\hbar})\hat{b}\hat{c}\,. \end{gathered} \tag{25}$$

The commutation relations of the usual quantum matrices for a real deformation parameter [31] can be obtained by the substitution $\mathrm{e}^\hbar \mapsto q$. The central and invariant square $\hat{l}^2$ of the quantum four-length is given by the quantum determinant

$$\hat{l}^2 := \hat{a}\hat{d} - \mathrm{e}^\hbar \hat{b}\hat{c}\,. \tag{26}$$

Quantum Euclidean four-space carries by construction a representation of the quantum orthogonal algebra $\mathcal{U}_\hbar(\mathrm{so}_4)$, which is, analogous to the undeformed case, the tensor algebra of two copies of $\mathcal{U}_\hbar(\mathrm{sl}_2)$, $\mathcal{U}_\hbar(\mathrm{so}_4) = \mathcal{U}_\hbar(\mathrm{sl}_2) \hat{\otimes} \mathcal{U}_\hbar(\mathrm{sl}_2)$. This shows that the isomorphism β from the orthogonal algebra to the quantum orthogonal algebra of theorem 1 is simply given by the tensor square of the isomorphism $\alpha : \mathcal{U}(\mathrm{sl}_2)[[\hbar]] \to \mathcal{U}_\hbar(\mathrm{sl}_2)$, which was constructed in the last section,

$$\beta : \mathcal{U}(\mathrm{so}_4)[[\hbar]] \xrightarrow{\cong} \mathcal{U}_\hbar(\mathrm{so}_4)\,, \quad \text{where} \quad \beta := \alpha \otimes \alpha\,. \tag{27}$$

In order to calculate the symmetry preserving ordering prescription we again need to reduce the deformed and the undeformed Euclidean space into their irreducible subrepresentations. This time the space algebras each possess a nontrivial invariant element, l^2 and $\hat{l}^2$, so the irreducible subspaces are degenerate. More precisely, every highest weight vector of $M_\hbar(2)$ is of the form $\hat{z}\hat{d}^{2j}$ for $2j \in \mathbb{N}_0$, where $\hat{z} \in \mathbb{C}[\hat{l}^2] \subset M_\hbar(2)$, the weight being (j, j). Let us denote by $\{\hat{T}^{(j,j)}_{mm'}\}$ the basis of the irreducible (j, j)-subrepresentation of $M_\hbar(2)$ which is generated by $\hat{d}^{2j}$ and by $\{T^{(j,j)}_{mm'}\}$ the according basis of $M(2)$ generated by d^{2j}.

If we want the symmetry preserving ordering prescription φ additionally to preserve the degree of the monomials, then φ must identify these bases, $\varphi(T^{(j,j)}_{mm'}) = \hat{T}^{(j,j)}_{mm'}$. Moreover, as l^2 and $\hat{l}^2$ are the only invariant elements of degree 2, we must have $\varphi(l^2) \sim \hat{l}^2$. For convenience we choose the proportionality constant to be 1. Observing that since l^2 is invariant we have $\varphi(l^{2n}x) = \varphi(l^{2n})\varphi(x) = \hat{l}^{2n}\varphi(x)$ for all $x \in M(2)$, we conclude that the symmetry preserving ordering prescription must be defined as

$$\varphi(l^{2n} T^{(j,j)}_{mm'}) = \hat{l}^{2n} \hat{T}^{(j,j)}_{mm'}\,. \tag{28}$$

We now want to express this ordering prescription in terms of the normal ordered (Poincaré–Birkhoff–Witt) bases. Towards this end we need to expand the irreducible basis $\hat{T}^{(j,j)}_{mm'}$ in terms of the normal ordered basis, and vice versa. Our starting point will be the multiplication map of $M_\hbar(2)$ in terms of the irreducible

basis, which has been calculated explicitly in [30],

$$\hat{T}^{(j_1,j_1)}_{m_1 m_1'} \hat{T}^{(j_2,j_2)}_{m_2 m_2'} = \sum_{j,m,m'} \left(\begin{smallmatrix} j_1 & j_2 \\ m_1 & m_2 \end{smallmatrix} \middle| \begin{smallmatrix} j \\ m \end{smallmatrix} \right)_q \left(\begin{smallmatrix} j_1 & j_2 \\ m_1' & m_2' \end{smallmatrix} \middle| \begin{smallmatrix} j \\ m' \end{smallmatrix} \right)_q \hat{l}^{2(j_1+j_2-j)}\, \hat{T}^{(j,j)}_{m,m'}\,, \tag{29}$$

where the expressions in parentheses denote the $\hbar$-adic quantum Clebsch–Gordan coefficients of $\mathcal{U}_\hbar(\mathrm{sl}_2)$, which can be obtained from the q-deformed Clebsch–Gordan coefficient [27] by the substitution $q \mapsto \mathrm{e}^\hbar$. Observing, that

$$\hat{T}^{(j,j)}_{-j,-j} = \hat{a}^{2j}\,, \quad \hat{T}^{(j,j)}_{-j,j} = \hat{b}^{2j}\,, \quad \hat{T}^{(j,j)}_{j,-j} = \hat{c}^{2j}\,, \quad \hat{T}^{(j,j)}_{j,j} = \hat{d}^{2j} \tag{30}$$

we get from Eq. (29)

$$\begin{aligned} \hat{a}^{2n_a}\hat{b}^{2n_b} &= \begin{bmatrix} 2n_a+2n_b \\ 2n_b \end{bmatrix}_{q^{-2}}^{-\frac{1}{2}} \hat{T}^{(n_a+n_b,n_a+n_b)}_{-n_a-n_b,-n_a+n_b} \\ \hat{c}^{2n_c}\hat{d}^{2n_d} &= \begin{bmatrix} 2n_c+2n_d \\ 2n_c \end{bmatrix}_{q^{-2}}^{-\frac{1}{2}} \hat{T}^{(n_c+n_d,n_c+n_c)}_{n_c+n_d,-n_c+n_d}\,. \end{aligned} \tag{31}$$

Multiplying these two monomials by Eq. (29) we get an expression of the normal ordered basis in terms of the reduced basis,

$$\begin{aligned} \hat{a}^{2n_a}\hat{b}^{2n_b}\hat{c}^{2n_c}\hat{d}^{2n_d} = \sum_j & \left(\begin{smallmatrix} n_a+n_b & n_c+n_d \\ -n_a-n_b & n_c+n_d \end{smallmatrix} \middle| \begin{smallmatrix} j \\ n_1 \end{smallmatrix} \right)_q \left(\begin{smallmatrix} n_a+n_b & n_c+n_d \\ -n_a+n_b & -n_c+n_d \end{smallmatrix} \middle| \begin{smallmatrix} j \\ n_2 \end{smallmatrix} \right)_q \\ & \times \begin{bmatrix} 2n_a+2n_b \\ 2n_b \end{bmatrix}_{q^{-2}}^{-\frac{1}{2}} \begin{bmatrix} 2n_c+2n_d \\ 2n_c \end{bmatrix}_{q^{-2}}^{-\frac{1}{2}} \hat{l}^{2(n-j)}\, \hat{T}^{(j,j)}_{n_1 n_2} \end{aligned} \tag{32}$$

for all $n_a, n_b, n_c, n_d \in \frac{1}{2}\mathbb{N}_0$, where $n_1 := -n_a - n_b + n_c + n_d$, $n_2 := -n_a + n_b - n_c + n_d$, and $n := n_a + n_b + n_c + n_d$. Using the orthogonality of the quantum Clebsch–Gordan coefficients Eq. (32) can be inverted,

$$\begin{aligned} \hat{l}^{2(n-j)}\, \hat{T}^{(j,j)}_{n_1 n_2} = \sum_k & \left(\begin{smallmatrix} n_a+n_b & n_c+n_d \\ -n_a-n_b & n_c+n_d \end{smallmatrix} \middle| \begin{smallmatrix} j \\ n_1 \end{smallmatrix} \right)_q^{-1} \left(\begin{smallmatrix} n_a+n_b & n_c+n_d \\ -n_a+n_b+k & -n_c+n_d-k \end{smallmatrix} \middle| \begin{smallmatrix} j \\ n_2 \end{smallmatrix} \right)_q \\ & \times \begin{bmatrix} 2n_a+2n_b \\ 2n_b+k \end{bmatrix}_{q^{-2}}^{\frac{1}{2}} \begin{bmatrix} 2n_c+2n_d \\ 2n_c+k \end{bmatrix}_{q^{-2}}^{\frac{1}{2}} \hat{a}^{2n_a-k}\hat{b}^{2n_b+k}\hat{c}^{2n_c+k}\hat{d}^{2n_d-k}\,. \end{aligned} \tag{33}$$

Applying the ordering prescription (28) to the undeformed limit of Eq. (32) and inserting Eq. (33) we finally obtain

$$\begin{aligned} \varphi(a^{2n_a}b^{2n_b}c^{2n_c}d^{2n_d}) = \sum_{j,k} & \frac{\left(\begin{smallmatrix} n_a+n_b & n_c+n_d \\ -n_a-n_b & n_c+n_d \end{smallmatrix} \middle| \begin{smallmatrix} j \\ n_1 \end{smallmatrix} \right) \begin{bmatrix} 2n_a+2n_b \\ 2n_b \end{bmatrix}_{q^{-2}}^{\frac{1}{2}} \begin{bmatrix} 2n_c+2n_d \\ 2n_c \end{bmatrix}_{q^{-2}}^{\frac{1}{2}}}{\left(\begin{smallmatrix} n_a+n_b & n_c+n_d \\ -n_a-n_b & n_c+n_d \end{smallmatrix} \middle| \begin{smallmatrix} j \\ n_1 \end{smallmatrix} \right)_q \binom{2n_a+2n_b}{2n_b+k}^{\frac{1}{2}} \binom{2n_c+2n_d}{2n_c+k}^{\frac{1}{2}}} \\ & \times \left(\begin{smallmatrix} n_a+n_b & n_c+n_d \\ -n_a+n_b & -n_c+n_d \end{smallmatrix} \middle| \begin{smallmatrix} j \\ n_2 \end{smallmatrix} \right) \left(\begin{smallmatrix} n_a+n_b & n_c+n_d \\ -n_a+n_b+k & -n_c+n_d-k \end{smallmatrix} \middle| \begin{smallmatrix} j \\ n_2 \end{smallmatrix} \right)_q \\ & \times \hat{a}^{2n_a-k}\hat{b}^{2n_b+k}\hat{c}^{2n_c+k}\hat{d}^{2n_d-k}\,, \end{aligned} \tag{34}$$

which is the desired expression for the symmetry preserving ordering prescription in terms of the normal ordered bases.

Via the q-binomial and q-Clebsch–Gordan coefficients, the dependence on the deformation parameter $\hbar$ is contained in basic hypergeometric series. In principle, the expansion of Eq. (34) in powers of $\hbar$ could be expressed in terms of combinatorial partition functions. Since there are no general number theoretic formulas for partition functions, this would not be of much practical value. Trying to calculate the coefficients of the $\hbar$-expansion explicitly, one would quickly learn that basic hypergeometric functions are the more efficient way to deal with partitions. Therefore, Eq. (34) is probably already the form which is best suited for applications. Moreover, using Eq. (34) as generating functional, the $\hbar$-expansion can be done by computer algebra.

3 Conclusion

In this paper perturbative deformations of symmetry structures on noncommutative spaces were studied. It was shown that the rigidity of symmetry algebras extends to rigidity of the action of the symmetry on the space. This result applies to all spaces with a symmetry given by a semi-simple Lie algebra and, hence, comprises most spaces which are of interest in physics. The generality of the results may be surprising at first sight: Even if a formal deformation of the symmetry structure looks extremely complicated, it is always isomorphic to the undeformed symmetry. But one has to keep in mind that the class of isomorphisms between rings over formal power series is very large. In general, the isomorphisms between the deformed and undeformed symmetry structure will not make numerical sense for a particular value of the deformation parameter. For the physical interpretation, this is not necessarily a problem as long as one stays within the realm of perturbation theory. This situation is not much worse than for ordinary quantum field theory where true convergence of loop expansions cannot be obtained easily, if at all.

The results obtained here can be applied to the construction of gauge theories on noncommutative spaces [3, 4] which is situated entirely in the realm of formal power series. In this context, the rigidity of symmetry structures has interesting implications for the construction of invariants, which would have to appear in Lagrangians. Consider the deformation of a space with a deformed symmetry structure such as quantum Euclidean space. If the star product is implemented with the symmetry preserving ordering prescription, then invariance with respect to the deformed symmetry is the same as invariance with respect to the undeformed symmetry, since the symmetry preserving ordering prescription maps invariants to invariants. Moreover, invariants can be constructed using the quantum metric.

This may not look surprising but let us illustrate with an example what can go wrong. Since the quantum plane does not have nontrivial invariants we take quantum Euclidean four-space. If the star product were realized by normal

ordering the quantum Euclidean four-length would be given by

$$a \star d - e^{\hbar} b \star c = \varphi^{-1}_{\text{normal}}(\hat{a}\hat{d} - e^{\hbar}\hat{b}\hat{c}) = ad - e^{\hbar} bc \tag{35}$$

which is not the invariant $ad - cd = x_0^2 + x_1^2 + x_2^2 + x_3^2$. In contrast, the ordering prescription given by formula Eq. (34) yields $\varphi(ad) = \hat{a}\hat{b}$ and $\varphi(bc) = e^{\hbar}\hat{b}\hat{c}$. Hence $\varphi^{-1}(\hat{a}\hat{d} - e^{\hbar}\hat{b}\hat{c}) = ad - bc$, as claimed. While it would be simple to ad-hoc modify the normal ordering in such a way that the quadratic invariant is preserved, preserving invariants of all orders would be difficult.

References

[1] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *JHEP* **09** (1999) 032, `hep-th/9908142`.

[2] M. Gerstenhaber, "On the deformation of rings and algebras," *Ann. Math.* **79** (1964) 59–103.

[3] J. Madore, S. Schraml, P. Schupp, and J. Wess, "Gauge theory on noncommutative spaces," *Eur. Phys. J.* **C16** (2000) 161–167, `hep-th/0001203`.

[4] B. Jurco, L. Moller, S. Schraml, P. Schupp, and J. Wess, "Construction of non-Abelian gauge theories on noncommutative spaces," *Eur. Phys. J.* **C21** (2001) 383–388, `hep-th/0104153`.

[5] M. R. Douglas and N. A. Nekrasov, "Noncommutative field theory," *Rev. Mod. Phys.* **73** (2001) 977–1029, `hep-th/0106048`.

[6] R. J. Szabo, "Quantum field theory on noncommutative spaces," *Phys. Rept.* **378** (2003) 207–299, `hep-th/0109162`.

[7] X. Calmet, B. Jurco, P. Schupp, J. Wess, and M. Wohlgenannt, "The standard model on non-commutative space-time," *Eur. Phys. J.* **C23** (2002) 363–376, `hep-ph/0111115`.

[8] E. O. Iltan, "The noncommutative effects on the dipole moments of fermions in the standard model," *JHEP* **05** (2003) 065, `hep-ph/0304097`.

[9] P. Minkowski, P. Schupp, and J. Trampetic, "Non-commutative '*-charge radius' and '*-dipole moment' of the neutrino," `hep-th/0302175`.

[10] P. Schupp, J. Trampetic, J. Wess, and G. Raffelt, "The photon neutrino interaction in non-commutative gauge field theory and astrophysical bounds," `hep-ph/0212292`.

[11] **OPAL** Collaboration, G. Abbiendi *et al.*, "Test of non-commutative QED in the process $e^+e^- \to \gamma\gamma$ at LEP," hep-ex/0303035.

[12] I. Hinchliffe and N. Kersting, "Review of the phenomenology of noncommutative geometry," hep-ph/0205040.

[13] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, "Noncommutative perturbative dynamics," *JHEP* **02** (2000) 020, hep-th/9912072.

[14] A. Matusis, L. Susskind, and N. Toumbas, "The IR/UV connection in the non-commutative gauge theories," *JHEP* **12** (2000) 002, hep-th/0002075.

[15] L. Alvarez-Gaume and M. A. Vazquez-Mozo, "General properties of noncommutative field theories," hep-th/0305093.

[16] D. J. Bird *et al.*, "Detection of a cosmic ray with measured energy well beyond the expected spectral cutoff due to comic microwave radiation," *Astrophys. J.* **441** (1995) 144–150.

[17] G. Amelino-Camelia, "Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale," *Int. J. Mod. Phys.* **D11** (2002) 35–60, gr-qc/0012051.

[18] G. Amelino-Camelia, "Doubly-Special Relativity: First Results and Key Open Problems," *Int. J. Mod. Phys.* **D11** (2002) 1643, gr-qc/0210063.

[19] A. Agostini, G. Amelino-Camelia, and F. D'Andrea, "Hopf-algebra description of noncommutative-spacetime symmetries," hep-th/0306013.

[20] Y. I. Manin, *Quantum Groups and Non-Commutative Geometry.* Centre de Recherche Mathématiques, Montréal, 1988.

[21] L. D. Faddeev, N. Y. Reshetikhin, and L. A. Takhtajan, "Quantization of Lie Groups and Lie Algebras," *Leningrad Math. J.* **1** (1990) 193–225.

[22] U. Carow-Watamura, M. Schlieker, M. Scholl, and S. Watamura, "Tensor Representation of the Quantum Group $SL_q(2)$ and Quantum Minkowski Space," *Z. Phys.* **C48** (1990) 159–166.

[23] V. G. Drinfeld, "Hopf algebras and the quantum Yang-Baxter equation," *Sov. Math. Dokl.* **32** (1985) 254–258.

[24] M. Jimbo, "A q-analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation," *Lett. Math. Phys.* **10** (1985) 63–69.

[25] V. G. Drinfeld, "On almost cocommutative Hopf algebras," *Leningrad Math. J.* **1** (1990), no. 2, 321–342.

[26] V. G. Drinfeld, “Quasi-Hopf Algebras,” *Leningrad Math. J.* **1** (1990), no. 6, 1419–1457.

[27] A. Klymik and K. Schmüdgen, *Quantum Groups and Their Representations*. Springer, 1997.

[28] C. Kassel, *Quantum Groups*. Springer, 1994.

[29] E. K. Sklyanin, “On an Algebra Generated by Quadratic Relations,” *Uspekhi Mat. Nauk* **40** (1985) 214.

[30] C. Blohmann, “Covariant realization of quantum spaces as star products by Drinfeld twists,” *J. Math. Phys.* **44** (2003) 4736–4755, `math.qa/0209180`.

[31] S. L. Woronowicz, “Twisted SU(2) group: An Example of a noncommutative differential calculus,” *Publ. Res. Inst. Math. Sci. Kyoto* **23** (1987) 117–181.